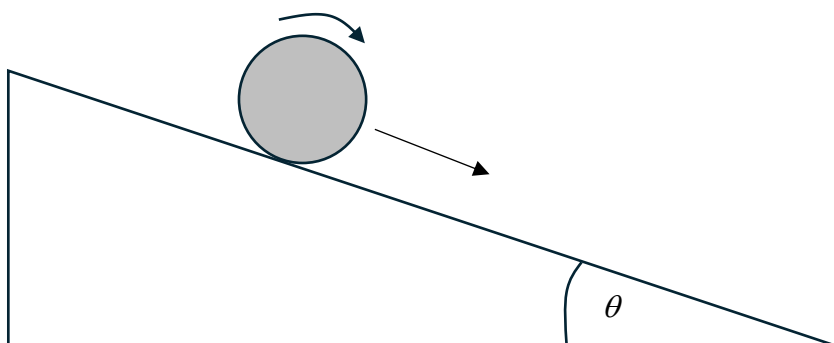


Rolling down an inclined plane without slipping

A solid body of mass M and radius R has moment of inertia $I = bMR^2$ where b is a number depending on the shape of the body. The body rolls without slipping down a plane inclined at an angle θ to the horizontal.



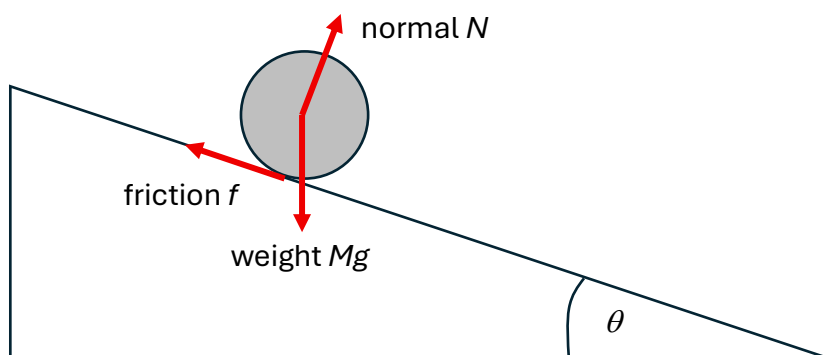
- Draw the forces on the body.
- Determine the acceleration of the center of mass.
- Determine that for a cylinder ($b = \frac{1}{2}$) and $\theta = 30^\circ$ the minimum coefficient of friction between the body and the plane needed to ensure no slipping is about 0.2.
- For a cylinder and $\theta = 30^\circ$ the coefficient of static friction is 0.40. Determine the frictional force on the cylinder in terms of its weight.
- The center of mass has been lowered by a vertical distance h after starting from rest.

Calculate

- the speed of the center of mass,
 - the angular momentum of the body,
 - the time to fall the vertical distance h .
- (f) Verify the relation $\tau = \frac{\Delta L}{\Delta t}$ using the results in (e) (i) and (e) (ii).
- (g) The body now climbs **up** the inclined plane having been given an initial angular speed.
- State and explain how the free body diagram in (a) changes, if at all.
 - Determine the linear deceleration of a cylinder going up an inclined plane with $\theta = 30^\circ$ without slipping.

Answers

(a)



(b) We take torques about the center of mass. Only the frictional force has a torque. The torque is fR . Hence

$$fR = I\alpha \quad (1.1)$$

where α is the angular acceleration.

Because there is no slipping, $\alpha = \frac{a}{R}$ where a is the linear acceleration of the center mass. Newton's second law says

$$mg \sin \theta - f = Ma \quad (1.2)$$

From (1.1) we find

$$f = \frac{I\alpha}{R} = \frac{Ia}{R^2} = \frac{bMR^2a}{R^2} = bMa \quad (1.3)$$

Substituting (1.3) in (1.2) we get

$$Mg \sin \theta - bMa = Ma \quad (1.4)$$

i.e. $Mg \sin \theta = (1 + b)Ma$ and so

$$(1.5)$$

$$a = \frac{g \sin \theta}{1 + b} \quad (1.6)$$

Hence

$$f = \frac{b}{1 + b} Mg \sin \theta \quad (1.7)$$

- (c) To find the **minimum** coefficient of static friction needed we must use the **maximum** frictional force that can develop between the body and the plane. In that case

$$f = \mu_{\min} N = \mu_{\min} Mg \cos \theta \quad (1.8)$$

From (1.3) and (1.5) we get $bM \frac{g \sin \theta}{1+b} = \mu_{\min} Mg \cos \theta$ and hence

$$\mu_{\min} = \frac{b}{1+b} \tan \theta \quad (1.9)$$

For a cylinder $b = \frac{1}{2}$ and for an angle $\theta = 30^\circ$, $\mu_{\min} = \frac{\frac{1}{2}}{1+\frac{1}{2}} \tan 30^\circ = 0.192$. This means

that any value of $\mu = 0.192$ and higher will allow the body to roll without slipping. A value of μ less than 0.192 will result in slipping.

- (d) The coefficient of friction is larger than the minimum of 0.192 and so the body can roll without slipping. From (1.7) the frictional force will be the same as before,

$$f = \frac{b}{1+b} Mg \sin \theta = \frac{\frac{1}{2}}{1+\frac{1}{2}} Mg \times \frac{1}{2} = \frac{1}{6} Mg$$

- (e)

- (i) Conservation of energy gives

$$Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 \quad (1.10)$$

i.e. $Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} bMR^2 \frac{v^2}{R^2}$. This simplifies to $2gh = (1+b)v^2$,

i.e.

$$v = \sqrt{\frac{2gh}{1+b}} \quad (1.11)$$

- (ii) $v^2 = \frac{2gh}{1+b}$, hence $\omega = \frac{v}{R} \Rightarrow \omega = \sqrt{\frac{2gh}{R^2(1+b)}}$. Hence $L = I\omega = bMR^2 \sqrt{\frac{2gh}{R^2(1+b)}}$

hence

$$L = bMR\sqrt{\frac{2gh}{1+b}} \quad (1.12)$$

$$(iii) \quad v = at \Rightarrow \sqrt{\frac{2gh}{1+b}} = \frac{g \sin \theta}{1+b} t, \text{ hence}$$

$$t = \frac{1}{\sin \theta} \sqrt{\frac{2(1+b)h}{g}} \quad (1.13)$$

$$(f) \quad \Delta L = bMR\sqrt{\frac{2gh}{1+b}} \text{ and } \Delta t = \frac{1}{\sin \theta} \sqrt{\frac{2(1+b)h}{g}}. \text{ Then } \frac{\Delta L}{\Delta t} = \frac{bMR\sqrt{\frac{2gh}{1+b}}}{\frac{1}{\sin \theta} \sqrt{\frac{2(1+b)h}{g}}} = \frac{b}{1+b} MgR \sin \theta$$

This is just fR i.e. the torque.

(g)

(i) The free body diagram will not change. The frictional force provides the torque that will now stop the body after moving some distance up the plane.

(ii) (1.2) is $f - Mg \sin \theta = Ma$ where a is the acceleration which will now come out negative since the body will decelerate. The frictional force is still given by (1.7) i.e.

$$f = \frac{b}{1+b} Mg \sin \theta \text{ and so}$$

$$\frac{b}{1+b} Mg \sin \theta - Mg \sin \theta = Ma$$

Hence $a = -\frac{g \sin \theta}{1+b}$ (numerically the same as before). This evaluates to

$$a = -\frac{g \sin \theta}{1+b} = -\frac{9.8 \times 0.5}{1.5} = -3.3 \text{ m s}^{-2}.$$